The last of these not only has remarkably small coefficients but also has a root more nearly equal to $\pi$ than we had any right to expect, namely $x = 3.1415925$.

The last four equations may be combined to satisfy various further conditions. For example, $F_1 - 2D_1$ gives us the third degree equation

$$41x^3 - 87x^2 - 89x - 133 = 0.$$

Or we may obtain an equation making a still closer approximation. Thus, $3D_1 - F_1$ gives the equation

$$4x^4 - 72x^3 + 125x^2 + 135x + 185 = 0,$$

which has a root equal to 3.1415926557, the value of $\pi$ being 3.1415926536. Or various other conditions might be imposed.

For the degree of accuracy used here the computation is rather laborious. If, however, only a single third degree equation is desired and not more than four or five significant figures in the root, the work is not long, especially if the powers of $x$ at the beginning are found by logarithms.

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# NAPIER'S LOGARITHMIC CONCEPT: A REPLY.

By FLORIAN CAJORI.

In the *Mathematical Gazette* of May, 1915, page 78, Professor H. S. Carslaw quotes the following passage from my article: "A History of the Exponential and Logarithmic Concepts," in the AMERICAN MATHEMATICAL MONTHLY of January, 1913, page 7:

> Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:
>
> $$\begin{array}{cccccc} v, & v\left(1-\frac{1}{v}\right), & v\left(1-\frac{1}{v}\right)^2, & \cdots, & v\left(1-\frac{1}{v}\right)^n, & \cdots \\ 0, & 1, & 2, & \cdots, & n, & \cdots. \end{array}$$
>
> The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log [10^7(1 - 10^{-7})^n] = n$, where $n = 0, 1, 2, \cdots$.

Professor Carslaw says: "This statement is incorrect. In Napier's Tables the logarithm of $(10^7 - 1)$ is not 1. It lies between 1 and 1.0000001, and he takes it as the mean between these two numbers, namely 1.00000005."

In reply to this I desire to make the following remarks: (1) In my article I did not explain at all Napier's *computation*; I aimed to explain his logarithmic *concept*. Napier's *theory* rests on the establishment of a one-to-one correspondence between the terms of a geometric series and the terms of an arithmetic series.[1] But, *it is not possible to write down two such series which represent exactly the numbers arising in Napier's computations*. Professor Carslaw himself admits that, in Napier's computations, "the numbers are not exactly in geometrical

[1] See Napier's *Constructio* (Macdonald's edition), page 19.

progression" (p. 82). Professor Carslaw remarks also that "when he (Napier) comes to define the term *logarithm,* he takes a new point of view altogether, and, though his logarithms nearly agree with those defined above, they do not do so absolutely."

I wrote down the two series quoted above, believing that they exhibited, all things considered, Napier's *theory* in its simplest and truest light. Professor Carslaw says that my exposition is "incorrect."

(2) I claim that the two series exhibit correctly the meaning of the word *logarithm*—"the number of the ratios."[1] The logarithm $n$ indicates the number of the ratios in the antilogarithm $v(1 - 1/v)^n$.

(3) I claim further that even according to Napier's computations, as explained in his *Constructio,* page 21, my statement is not "incorrect." When speaking of upper and lower limits of the logarithm of a given sine, Napier says:

> "The limits themselves differing insensibly, they or anything between them may be taken as the true logarithm. Thus, in the above example, the logarithm of the sine 9999999 was found to be either 1.0000000 or 1.00000010, or best of all 1.00000005. For since the limits themselves, 1.0000000 and 1.0000001, differ from each other by an insensible fraction like 1/10000000, therefore they and whatever is between them will differ still less from the true logarithm lying between these limits."

Napier here admits 1 as a value of log $(10^7 - 1)$ which differs "insensibly" from the true value. The true value demanded by his theory of moving points is neither 1 nor 1.00000005; it is a little greater than the latter.

(4) I cheerfully admit that taking log $(10^7 - 1) = 1.00000005$ would have been "best of all" for the purpose of exhibiting somewhat more closely the results of Napier's *computation.* The first 101 numbers from $10^7$ to 9999900.0004950, found in Napier's "First Table" in the *Constructio* (pp. 13, 22), are in geometric progression; Napier assigns multiples of 1.00000005 as their respective logarithms. But for reasons mentioned above, taking 1.00000005 as the first term of an arithmetical progression does not accurately reproduce all of Napier's results of computation. Moreover, 1.00000005 would have been less satisfactory than 1 for the purpose of exhibiting the notion of "the number of the ratios." I still think that in a very brief exposition, such as I gave in the article quoted, Napier's logarithmic *concept* is made plainer by the two series as I gave them, than by the introduction of eight-place decimals in each term of the arithmetic series, which complicate the bird's-eye view, without showing with absolute accuracy Napier's tabular results.[2] That my exposition is "incorrect" I cannot admit.

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[1] Professor Carslaw and I interpret the word "logarithm" as meaning "the number of the ratios." Some writers prefer another derivation, namely, "ratio-number" or "number associated with ratio."

[2] I seize this opportunity to state that the two Latin definitions of logarithms quoted in my article in the AMERICAN MATHEMATICAL MONTHLY, Vol. 20, 1913, page 7, are the phrasings given by Briggs and not by Napier.